Front page

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A Method to Compute Neutrino Oscillation Probabilities for Arbitrary Time-Dependent Hamiltonians

Bachelor Thesis

Written by *Julie H. Gabrielsen*

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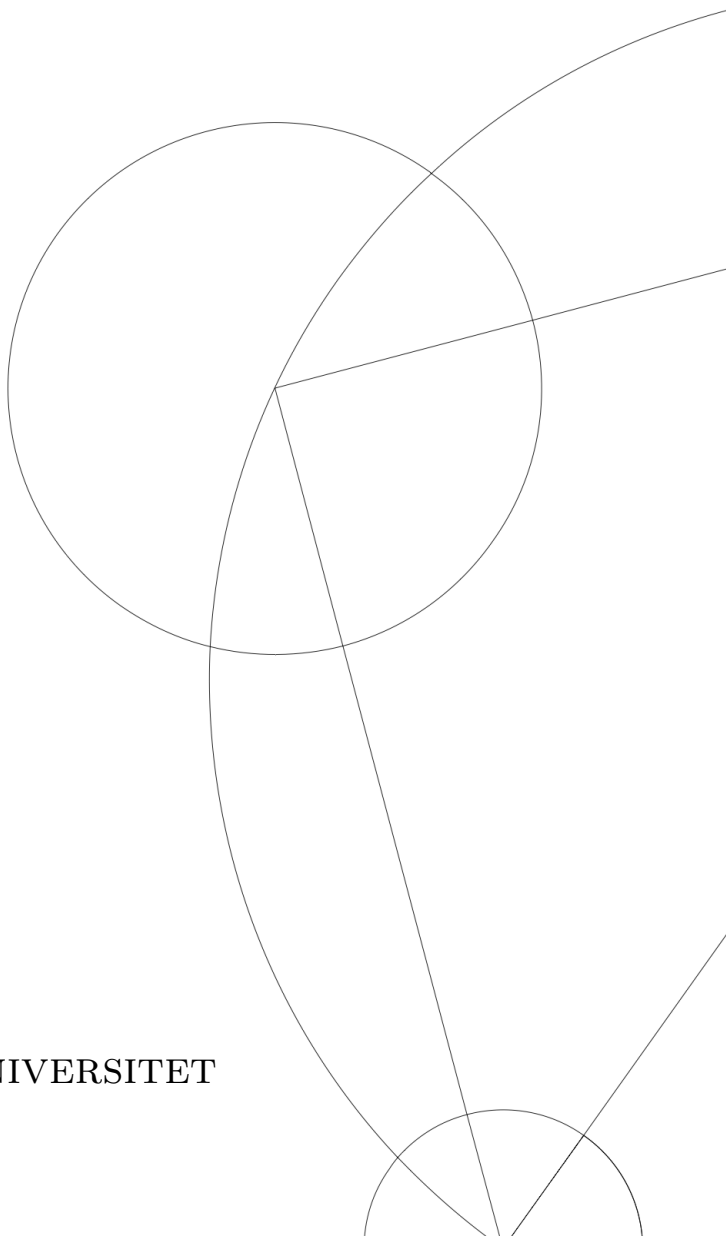

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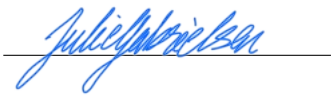
Signature 

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Abstract

In this project, a method was developed to compute neutrino oscillation probabilities for arbitrary two- and three-neutrino time-dependent Hamiltonians, without having to diagonalize the Hamiltonian. This new method consists of looking at very small intervals when computing the probability amplitude at each step.

When comparing the method to a known and exact solution for time-independent Hamiltonians, the new method gives exact results within the small intervals and over all very precise results with a small error. The method was also used to find the neutrino oscillation probability in cases with time-dependent Hamiltonians; going through the Earth, and through the Sun in the two-neutrino flavor case. These probabilities also behave as expected.

Contents

1 Introduction

The discovery of neutrino flavor transitions meant that neutrinos must have mass, and that the Standard Model, which orginally posited massless neutrinos, does not accurately describe them [9, 12]. As will be clear from the plots shown in Chapter 4, the neutrino flavor transition probability oscillates, and the flavor transitions are therefore called oscillations. Computing and measuring the neutrino oscillation probabilities can, among other things, tell us about the differences in flux of astrophysical neutrinos before they arrive at Earth, and the mass differences between the neutrinos - though the neutrino oscillations cannot be used to find the masses themselves [10].

When calculating the neutrino oscillation probabilities, the Hamiltonian that describes their propagation and the mixing between flavors typically has to be diagonalized in order to solve the Schrödinger equation that describes their time evolution. When working with three-flavor transitions or neutrinos moving through matter, the Hamiltonian can become quite complex, and the expression obtained from the diagonalization can therefore be so complex that they become practically unusable. This creates a need for approximations when writing analytical expressions for the probability, which often have associated errors at the percentage level [4].

Reference [4] introduces the so-called NOPE (NuOscProbExact) method. The NOPE method computes neutrino oscillation probabilities, without having to diagonalize the Hamiltonian, for systems of two and three neutrino flavors that uphold the following criteria:

1. The system must be closed so that the number of neutrinos across of flavors is always conserved.
2. The Hamiltonian must be time-independent.

The goal of this project is to further develop the NOPE method to make it work for time-dependent Hamiltonians, e.g., to compute the oscillation of neutrinos inside matter with a varying density profile, like inside the Earth or the Sun.

2 Theory of Neutrino Oscillations

2.1 Neutrino Oscillations

There are three known neutrino flavors, the electron neutrino (ν_e), the muon neutrino (ν_μ), and the tau neutrino (ν_τ), which match the three leptons. The flavor eigenstates are superpositions of the mass eigenstates ν_1, ν_2, and ν_3, with masses m_1, m_2, and m_3, [10]

$$\begin{pmatrix} |\nu_e\rangle \\ |\nu_\mu\rangle \\ |\nu_\tau\rangle \end{pmatrix} = \mathbb{U}_{\text{PMNS}} \begin{pmatrix} |\nu_1\rangle \\ |\nu_2\rangle \\ |\nu_3\rangle \end{pmatrix}, \tag{1}$$

where $\mathbb{U}_{\text{PMNS}}$ is the Pontecorvo-Maki-Nakagawa-Sakata (PMNS) lepton mixing matrix, parametrized in terms of three mixing angles, θ_{12}, θ_{23}, and θ_{13}, and one CP-violation phase, δ_{CP} [15]. Thus, the flavor states, $|\nu_\alpha\rangle$, where $\alpha = e, \mu, \tau$, can be written as

$$|\nu_\alpha\rangle = \sum_{i=1}^{3} \mathbb{U}^*_{\text{PMNS},\alpha\text{i}} |\nu_i\rangle. \tag{2}$$

In theory, it is possible to measure the flavor of a neutrino with $\nu_\alpha + l_\alpha^+ \to W$, where ν_α are the three neutrino flavors, and l_α^+ are the three corresponding charged leptons. Because of lepton number conservation, it would then be possible to know the flavor of the neutrino by knowing which lepton it interacts with. In practice, it is not possible to create enough neutrinos for this kind of reaction, so the following experiment can be put together. In this experiment, neutrinos are directed towards a large basin of water which is placed underground so as not to be disturbed by other neutrinos. This water will mainly consist of protons, and the inner walls of the basins will be covered with photo multipliers. The idea is then to make the neutrino undergo the decay $\nu_l + p \to l_\alpha^- + X$, where ν_α are the three flavors, l_α^- are the three corresponding anti-charged leptons, p is the proton and X represents hadrons. The exact composition of the final state of the interaction depends on the energy of the interacting neutrino and its flavor. In any case, the final-state particles initiate particle showers or are captured by the medium, emitting light that is picked up by photomultipliers.

Neutrinos can change flavor over time or, equivalently, distance, as it propagates from its point of creation. The probability that a neutrino created with a given flavor is detected later having a different flavor "oscillates" with $\frac{L}{E}$, where L is the distance and E is the energy of the neutrino. That being said, the potential does have an impact on the neutrino oscillation probabilities. This will be further described in Chapter 4.2.

Before neutrino oscillations were measured, it was believed that neutrinos were likely massless. However, the neutrino oscillation probability demands that at least two of neutrino mass eigenstates have nonzero mass, as we show below.

2.2 Two-Neutrino Oscillations in Vacuum

In the two-flavor approximation, we look at a system with only two neutrino flavors. The approximation is useful as many experimental setups are designed to be primarily sensitive to the transition between two flavors. It is easier to calculate the probability, since the mixing matrix is simpler in the two-flavor case, and does not include any CP-violation phase as it does in the three-flavor case (Eq. 13).

Below, we present the standard derivation of the two-neutrino oscillation probability in vacuum. The derivation within the NOPE method will be presented in Chapter 3.1.1.

2.2.1 Classical Derivation of the Two-Neutrino Oscillation Probability in Vacuum

In the two-flavor approximation, the leptonic mixing matrix is a 2×2 rotation matrix parametrized by a single mixing angle, θ, i.e.,

$$\mathbb{U}_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}. \tag{3}$$

The neutrino flavor eigenstates, $|\nu_\alpha\rangle$ and $|\nu_\beta\rangle$, are superpositions of the mass eigenstates, $|\nu_1\rangle$ and $|\nu_2\rangle$, i.e.,

$$\begin{pmatrix} |\nu_\alpha\rangle \\ |\nu_\beta\rangle \end{pmatrix} = \mathbb{U}_\theta \begin{pmatrix} |\nu_1\rangle \\ |\nu_2\rangle \end{pmatrix} = \begin{pmatrix} \cos\theta|\nu_1\rangle + \sin\theta|\nu_2\rangle \\ -\sin\theta|\nu_1\rangle \cos\theta|\nu_1\rangle \end{pmatrix}. \tag{4}$$

The mass eigenstates are eigenstates of the Hamiltonian and satisfy the Schrödinger equation

$$i\frac{\partial}{\partial t}|\nu_i\rangle = \mathbb{H}|\nu_i\rangle = E_i|\nu_i\rangle, \tag{5}$$

where E_i is the energy of $|\nu_i\rangle$. We start by assuming that the Hamiltonian is time-independent (as in oscillations in vacuum or in matter with uniform density), and, therefore, so is E_i, and the solution of the Schrödinger equation is

$$|\nu_i(t)\rangle = e^{-iE_i t}|\nu_i\rangle. \tag{6}$$

Since the neutrinos are relativistic (i.e., $E_i \gg m_i$), we can write their energy as

$$E_i = \sqrt{|\vec{p}|^2 + m_i^2} \simeq |\vec{p}| + \frac{m_i^2}{2|\vec{p}|} \simeq \frac{m_i^2}{2E}, \tag{7}$$

where it is used that the momentum, $\vec{p}$, is the same for all eigenstates and is therefore a global phase, and that neutrinos are relativistic so that E is the average energy of the eigenstates. In the mass basis, the Hamiltonian will be diagonal, with the entries being the energies:

$$\mathbb{H}_2^m = \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix} = \begin{pmatrix} \frac{m_1^2}{2E} & 0 \\ 0 & \frac{m_2^2}{2E} \end{pmatrix}. \tag{8}$$

Thus, the amplitude for the transition $\nu_\alpha \to \nu_\beta$ is

$$\begin{aligned} \mathcal{A}_{\nu_\alpha \to \nu_\beta} &= \langle \nu_\beta | \nu_\alpha(t) \rangle \\ &= (-\langle \nu_1| \sin\theta + \langle \nu_2| \cos\theta) e^{-i\mathbb{H}_2^m t} (\cos\theta|\nu_1\rangle + \sin\theta|\nu_2\rangle) \\ &= -\sin\theta\cos\theta e^{-i\frac{m_1^2}{2E}t} + \sin\theta\cos\theta e^{-i\frac{m_2^2}{2E}t}, \end{aligned} \tag{9}$$

And the neutrino oscillation probability is

$$\begin{aligned} P_{\nu_\alpha \to \nu_\beta} &= |\mathcal{A}_{\nu_\alpha \to \nu_\beta}|^2 \\ &= \sin^2\theta\cos^2\theta \left(e^{i\frac{m_1^2 - m_1^2}{2E}t} + e^{i\frac{m_2^2 - m_2^2}{2E}t} - e^{i\frac{m_1^2 - m_2^2}{2E}t} + e^{i\frac{m_2^2 - m_1^2}{2E}t} \right) \\ &= \sin^2\theta\cos^2\theta \left(2 - 2\cos\left(\frac{\Delta m^2}{2E} t \right) \right) \\ &= \sin^2(2\theta) \sin^2\left(\frac{\Delta m^2}{4E} L \right), \end{aligned} \tag{10}$$

for $\alpha \neq \beta$, where $\Delta m^2 \equiv m_2^2 - m_1^2$ is the mass-squared difference. At relativistic velocities, we can use the approximation $L \simeq t$ in natural units where the speed of light $c = 1$.

From CPT symmetry we find:

$$\begin{aligned} P_{\bar{\nu}_\alpha \to \bar{\nu}_\beta} &= P_{\nu_\beta \to \nu_\alpha} \\ &= |(\langle \nu_1| \cos\theta + \langle \nu_2| \sin\theta) e^{-i\mathbb{H}^m t}(-\sin\theta|\nu_1\rangle + \cos\theta|\nu_2\rangle)|^2 \\ &= P_{\nu_\alpha \to \nu_\beta}, \end{aligned} \tag{11}$$

which means that the probability is the same for neutrinos and antineutrinos.

2.3 Two-Neutrino Oscillations in Matter

When neutrinos propagate in matter, their oscillations are affected by coherent forward scattering. This scattering happens in matter in two different ways:

1. A ν_e particle exchanges a W boson with an electron. This interaction gives the potential energy $V_W = \sqrt{2} G_F N_e$, where G_F is the Fermi coupling constant, and N_e is the number of electrons per unit volume. For the $\bar{\nu}_e$, the potential is the same with opposite sign.
2. A neutrino of any flavor exchanges a Z boson with an electron, proton, or neutron. This interaction gives the potential energy $V_Z = -\frac{\sqrt{2}}{2} G_F N_e$.

Thus, for neutrinos propagating in matter, the Hamiltonian is

$$\mathbb{H}_2^M = \mathbb{H}_2^{\text{vac}} + V_W \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + V_Z \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{12}$$

where, because the potential due to neutral-current interactions affects all flavors equally, it introduces a global phase that does not affect the probability and that, therefore, we have discarded. The number density of electrons, N_e, is constant if the neutrinos propagate inside uniform matter; otherwise, it is a function of the position of the neutrino. In the former case, the Hamiltonian is time-independent, and we can solve the Schrödinger equation as before, but with eigenvalues modified by the matter potential. In the latter case, the Hamiltonian is time-dependent, and so finding the flavor state at a later time involves solving the Schrödinger equation for the specific scenario under consideration.

For derivation of the neutrino oscillation probability in matter in the two-flavor case, see Appendix A.

2.4 Three-Neutrino Oscillations in Vacuum

In the three-neutrino flavor case, the mixing matrix is the PMNS matrix, parametrized as

$$\mathbb{U}_{\text{PMNS}} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta_{CP}} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta_{CP}} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta_{CP}} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta_{CP}} & -c_{12}s_{23} - s_{12}c_{23}s13e^{i\delta_{CP}} & c_{23}c_{13} \end{pmatrix}, \tag{13}$$

where $c_{ij} \equiv \cos\theta_{ij}$ and $s_{ij} \equiv \sin\theta_{ij}$ [15]. The neutrino oscillation probability is

$$\begin{aligned} P_{\nu_\alpha \to \nu_\beta} =& \delta_{\alpha\beta} - 4 \sum_{i>j} \text{Re}\left(U^*_{0,\alpha i} U_{0,\beta i} U_{0,\alpha j} U^*_{0,\beta j}\right) \sin^2\left(\frac{\Delta m^2_{ij}}{2E} L\right) \\ &+ 2 \sum_{i>j} \text{Im}\left(U^*_{0,\alpha i} U_{0,\beta i} U_{0,\alpha j} U^*_{0,\beta j}\right) \sin\left(\frac{\Delta m^2_{ij}}{2E} L\right) \end{aligned} \tag{14}$$

For derivation of the three-neutrino oscillation probability in vacuum, see Appendix B.

3 The NOPE Method

3.1 Time-Independent Hamiltonians

In the classical derivation of the neutrino oscillation probability (Chapter 2.2.1, Appendices A and B), the diagonalization of the Hamiltonian is an essential step. Depending on the complexity of the Hamiltonian, this can be a difficult and time-consuming step, and, if one wishes to write analytical approximate expressions for the probability, they can be quite complicated and not illuminating. In the method developed by Ohlsson and Snellman [16, 17, 19, 20], the Hamiltonian does not need to be diagonalized, and the result from this method is an exact expression for the oscillation probability which is easier to interpret and therefore to implement without approximation. The method used in this project is further developed by M. Bustamante to work on all systems that are closed (so the number of neutrinos are conserved across the flavors) and that can be described using a time-independent Hamiltonian. [3, 4]

3.1.1 Derivation of the Two-Neutrino Oscillation Probability in Vacuum using the NOPE method

We start with the case of a time-independent 2×2 Hamiltonian, as in Ref. [4]. In the two-neutrino case, instead of diagonalizing the Hamiltonian, the method starts by expanding it as a sum of Pauli matrices, σ^k, as

$$\mathbb{H}_2 = h_0 \mathbb{1} + h_k \sigma^k, \tag{15}$$

where the h-coefficients can be found in Table 1.

The evolution operator is expanded using the exponential expansion of the Pauli matrices by implementing the Cayley-Hamilton theorem [1, 6, 22]. The Cayley-Hamilton theorem states that any analytical function of a $n \times n$ matrix can be written as a polynomial of degree $(n-1)$ in that matrix.

$$\begin{aligned} \mathbb{U}_2(L) &= e^{-i\mathbb{H}_2 L} \\ &= e^{-i(h_0\mathbb{1} + h_k\sigma^k)} \\ &= \cos(|h|L)\mathbb{1} - i h_k \sigma^k \frac{\sin(|h|L)}{|h|}, \end{aligned} \tag{16}$$

where $|h|^2 \equiv |h_1|^2 + |h_2|^2 + |h_3|^2$ and Euler's formula is generalized as $e^{\pm i a_k \sigma^2} = \cos(|a|) \pm i a_k \sigma^k \sin(|a|)$, using the identity of Pauli matrices. The part $e^{-i(h_0\mathbb{1})}$ is a global phase and is therefore disregarded, and $h_k\sigma^k$ expands to:

$$\begin{aligned} h_k\sigma^k &= h_1 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + h_2 \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + h_3 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ &= \begin{pmatrix} h_3 & h_1 - i h_2 \\ h_1 + i h_2 & -h_3 \end{pmatrix}. \end{aligned} \tag{17}$$

We can now find the probability of a neutrino to be detected with the same flavor with which it was produced, i.e., the neutrino survival probability:

h_0	$\frac{1}{2}((\mathbb{H}_2)_{11} + (\mathbb{H}_2)_{22})$
h_1	$\mathcal{R}((\mathbb{H}_2)_{12})$
h_2	$-\mathcal{I}((\mathbb{H}_2)_{12})$
h_3	$\frac{1}{2}((\mathbb{H}_2)_{11} - (\mathbb{H}_2)_{22})$

Table 1: Table of h-coefficients in two-flavor case. [4]

$$
\begin{aligned}
P_{\nu_\alpha \to \nu_\alpha}(L) &= |\langle \nu_\alpha | \mathbb{U}_2(L) | \nu_\alpha \rangle|^2 \\
&= \left| \cos(|h|L) - i \frac{\sin(|h|L)}{|h|} h_3 \right|^2 \\
&= 1 - \frac{|h_1|^2 + |h_2|^2}{|h|^2} \sin^2(|h|L).
\end{aligned} \tag{18}
$$

The associated transition probability is

$$
P_{\nu_\alpha \to \nu_\beta}(L) = 1 - P_{\nu_\alpha \to \nu_\alpha}(L) = \frac{|h_1|^2 + |h_2|^2}{|h|^2} \sin^2(|h|L). \tag{19}
$$

This is the general solution for time-independent Hamiltonians in the two-flavor case, and can therefore also be used in matter and non-standard interactions. Specifically, in vacuum we have the Hamiltonian

$$
\mathbb{H}^{\text{vac}}_{2,m} = \frac{\Delta m^2}{4E} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{20}
$$

In the flavor basis we then have

$$
\mathbb{H}^{\text{vac}}_{2,f} = \mathbb{U}_\theta \mathbb{H}^{\text{vac}}_{2,m} \mathbb{U}^\dagger_\theta = \frac{\Delta m^2}{4E} \begin{pmatrix} -\cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & \cos(2\theta) \end{pmatrix}. \tag{21}
$$

We can now find the h-coefficients via Table 1:

$$
|h_1|^2 = \left(\frac{\Delta m^2}{4E} \right)^2 \sin^2(2\theta), |h_2|^2 = 0, |h_3|^2 = \left(\frac{\Delta m^2}{4E} \right)^2 \cos^2(2\theta), |h| = \frac{\Delta m^2}{4E}. \tag{22}
$$

Inserting the coefficients in Eq. 19 we get

$$
P_{\nu_\alpha \to \nu_\beta}(L) = \sin^2(2\theta) \sin^2 \left(\frac{\Delta m^2}{4E} \right), \tag{23}
$$

which is the same expression we got from the classical derivation in Chapter 2.2.1.

In the three-flavor case the procedure is analogous, but the Hamiltonian, $\mathbb{H}_3$, is a 3×3 and it is expanded using Gell-Mann matrices. The h-coefficients in the three flavor case are in Table 2.

3.2 Expanding the NOPE Method for Time-Dependent Hamiltonians

For time-dependent Hamiltonians, the solutions to the Schrödinger equation gives the evolution operator

h_0	$\frac{1}{3}((\mathbb{H}_3)_{11} + (\mathbb{H}_3)_{22} + (\mathbb{H}_3)_{33})$
h_1	$\text{Re}((\mathbb{H}_3)_{12})$
h_2	$-\text{Im}((\mathbb{H}_3)_{12})$
h_3	$\frac{1}{2}((\mathbb{H}_3)_{11} + (\mathbb{H}_3)_{22})$
h_4	$\text{Re}((\mathbb{H}_3)_{13})$
h_5	$-\text{Im}((\mathbb{H}_3)_{13})$
h_6	$\text{Re}((\mathbb{H}_3)_{23})$
h_7	$-\text{Im}((\mathbb{H}_3)_{23})$
h_8	$\frac{\sqrt{3}}{6}((\mathbb{H}_3)_{11} + (\mathbb{H}_3)_{22} - 2(\mathbb{H}_3)_{33})$

Table 2: Table of h-coefficients in three flavor case. [4]

$$\mathbb{U}(t, t+\Delta) = \mathbb{T} e^{-i \int_t^{t+\Delta} \mathbb{H}_2(\tau) d\tau}, \tag{24}$$

where $\mathbb{T}$ is the time-ordering operator, which sorts the product according to when they take place, from lowest to highest time-value.

The first solution attempted was to directly solve the integral numerically, for a wide time interval Δ. We tried two different approaches: an adaptive integration routine (implementing in the SciPy `quad` function) and a solution using Gauss-Legendre quadrature, i.e.,

$$\mathbb{U}_\delta(t, t+\Delta) \approx e^{-i\xi \sum_{j=1}^{p} w_j \mathbb{H}_2(t_j)}, \tag{25}$$

where p is a fixed number of evaluation points at which the quadrature weights are computed, and $\lim_{p\to\infty} \mathbb{U}_\delta = \mathbb{U}$. In both cases, we started by ignoring the application of the time-ordering operator, since there is, in practice, no way to time-order the operations inside the single integral in Eq. 24 covering the entire time width Δ. As a result, both methods to approximate the integral failed to yield correct results when applied to time-dependent Hamiltonians. The reason is that, without the application of the time-ordering operator in Eq. 24, it can only accurately represent the time evolution driven by a time-independent Hamiltonian. In other words, for Eq. 24 to be valid over a wide time interval, the Hamiltonian evaluated at different times within it must commute with itself, which, in general, is not the cast for time-dependent Hamiltonians. (To accurately compute the time-evolution operator driven by a time-dependent Hamiltonian over a wide time interval using a form similar to Eq. 24, one would need to use the Magnus expansion [11].)

Therefore, we decided to adopt a different strategy to compute the evolution of the neutrino states. Rather than solving Eq. 24 over the entire, wide time interval Δ, we split the interval into multiple narrow intervals of equal width. Within each interval, we assume that the Hamiltonian is constant and compute the evolution operator using the NOPE method for a time-independent Hamiltonian, as introduced in Chapter 3.1. The evolution operator over the entire time window Δ is then the time-ordered product of the evolution operators of each time interval, i.e.,

$$\begin{aligned} \mathbb{U}_2(L) &= \prod_{i=1}^{N_{\text{int}}-1} U_2(l_i, l_{i+1}) \\ &= \prod_{i=1}^{N_{\text{int}}-1} \left(\cos(|h^{(i)}|\delta L_i)\mathbb{1} - i h_k \sigma^k \frac{\sin(|h| dL_i)}{|h|} \right), \end{aligned} \tag{26}$$

in the two-flavor case, where $|h^{(i)}|$ is the modulus of the SU(2)-expansion coefficients of H_2 evaluated within the i-th interval, of size $\delta L_i \equiv l_{i+1} - l_i$. Within the i-th interval, the Hamiltonian is taken to be constant, with its value evaluated at the center point of the interval, i.e., $H_2(l_i + (l_{i+1} - l_i)/2)$. Analogously, for the three-flavor case,

$$\begin{aligned} \mathbb{U}_3(L) &= \prod_{i=1}^{N_{\text{int}}-1} \mathbb{U}_3(dL) \\ &= \prod_{i=1}^{N_{\text{int}}-1} \left(\cos(|h| dL_i)\mathbb{1} - i h_k \lambda^k \frac{\sin(|h| dL_i)}{|h|} \right), \end{aligned} \tag{27}$$

in the three-neutrino flavor case, where $dL_i = l_i - l_{i-1}$ for i in the smaller interval.

The two-neutrino oscillation probability is then computed as

$$
\begin{aligned}
P_{\nu_\alpha \to \nu_\beta}(L) &= |\langle \nu_\beta | \mathbb{U}_2(L) | \nu_\alpha \rangle|^2 \\
&= \begin{pmatrix} 0 & 1 \end{pmatrix} \mathbb{U}_2(L) \begin{pmatrix} 0 \\ 1 \end{pmatrix}.
\end{aligned}
\tag{28}
$$

The three-neutrino oscillation probability is computed as

$$
P_{\nu_\alpha \to \nu_\beta}(L) = |\langle \nu_\beta | \mathbb{U}_3(L) | \nu_\alpha \rangle|^2. \tag{29}
$$

3.3 Implementation of the Expanded Method

Computing the two-neutrino oscillation probabilities for time-dependent Hamiltonians was done the same way as for the time-independent Hamiltonians with the addition of two functions to the NuOscProbExact [3] code: `probabilities_2nu_tslice` and `evolution_operator_2nu_tslice`. The function `evolution_operator_2nu_tslice` computes the evolution operator as in Eq. 26, while `probabilities_2nu_tslice` computes the neutrino oscillation probability as in Eq. 28 with the input being the Hamiltonian ($\mathbb{H}_2$), l_{initial}, l_{final} and N_{int}, the number of intervals.

Correspondingly two function were added to compute the three-neutrino oscillation probabilities: `probabilities_3nu_tslice` and `evolution_operator_3nu_tslice`. The function `evolution_operator_3nu_tslice` computes the evolution operator as in Eq. 27, while `probabilities_2nu_tslice` computes the neutrino oscillation probability as in Eq. 29 with the input being the Hamiltonian ($\mathbb{H}_3$), l_{initial}, l_{final} and N_{int}, the number of intervals.

4 Results

4.1 Two- and Three-Neutrino Oscillations in Vacuum

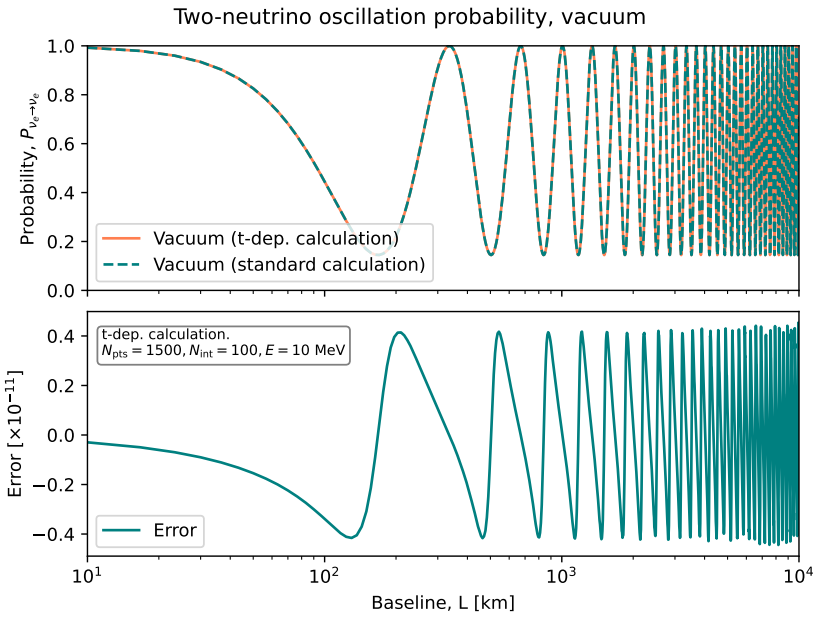

Figure 1: Two-neutrino survival probability computed using the methods introduced in Eqs. 26 and 28, as a function of distance in vacuum evaluated at $E = 10$ MeV, comparing it to the probability calculated using the standard formula (Eq. 10). The data consists of $N_{\text{pts}} = 1500$ and $N_{\text{int}} = 100$.

All plots in Chapter 4 are plotted with an amount of data points, N_{pts}. Since we want the probability as a function of L, we need to define a grid. This we do by defining N_{pts}, which is the amount of grid points from x_{initial} to x_{final}. In turn, $\frac{N_{\text{pts}}}{N_{\text{int}}}$ will also give us the size of the intervals, which will be used to approximate the probability at some $L = x_i$.

In order to find out how well the new method works, we compare the probability in vacuum computed using it vs. the probability computed using the standard probability expression, Eq. 10.

Figures 1 and 2 show the neutrino probabilities as a function and the error between them. Here, it can be seen that the error is small, of the order of 10^{-10} and grows with distance. This is expected as the neutrino oscillation probability oscillates with $\frac{L}{E}$, so a greater distance results in a smaller oscillation length, resulting in a larger uncertainty. The error is computed as

$$\Delta = \frac{P_{\text{std},\nu_e \to \nu_e} - P_{\text{comp},\nu_e \to \nu_e}}{P_{\text{std},\nu_e \to \nu_e}}. \tag{30}$$

A noticeable difference between Figs. 1 and 2 is that there is more structure to the probability in the three-flavor case, which makes sense as the three-neutrino oscillation probability (Eq. 54 consists of more sinusoidal functions than the three-neutrino oscillation probability (Eq. 10) because of there being one more Δm^2.

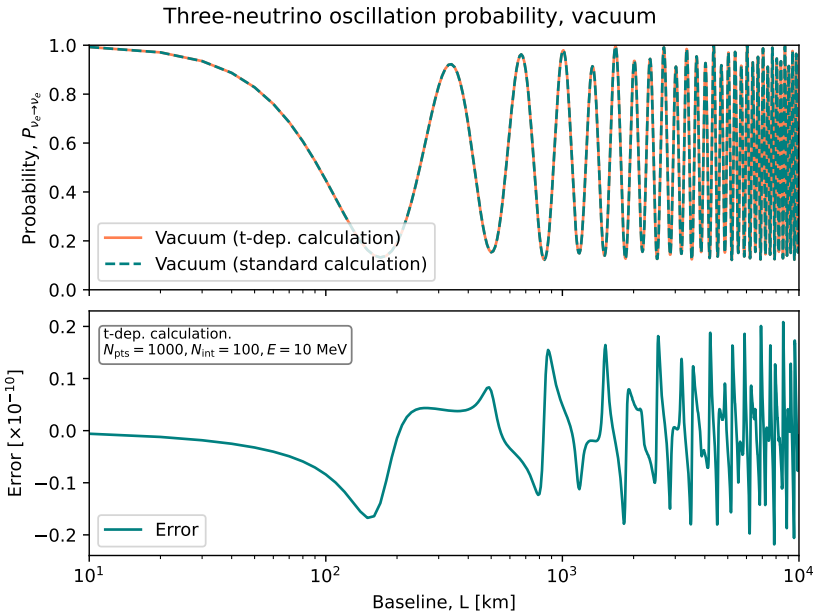

Figure 2: Three-neutrino survival probability computed using the methods introduced in Eqs. 27 and 29, as a function of distance in vacuum evaluated at $E = 10$ MeV, comparing it to the probability calculated using the standard formula (Eq. 54). The data consists of $N_{\text{pts}} = 1000$ and $N_{\text{int}} = 100$.

Figures 3 and 4 show a similar behavior for the neutrino survival probability in vacuum vs. energy as vs. distance in Figs. 1 and 2. In Figs. 3 and 4 it can be seen that the error is small, of the order of 10^{-11} and decreases with energy. Analogous to Figs. 1 and 2, a lower energy results in a smaller oscillation length, which results in a larger uncertainty.

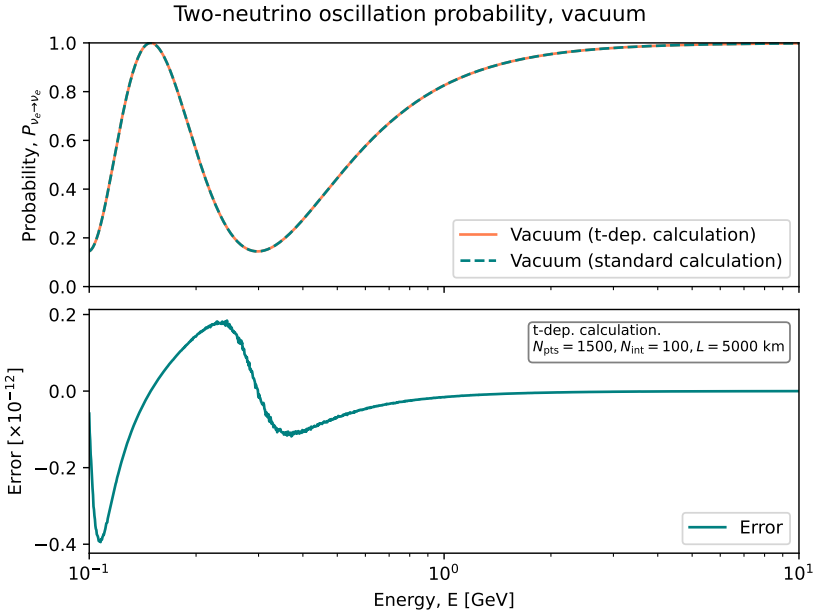

Figure 3: Two-neutrino survival probability computed using the methods introduced in Eqs. 26 and 28, as a function of energy in vacuum evaluated at $L = 5000$ km, comparing it to the probability in vacuum calculated using the standard formula (Eq. 10). The data consists of $N_{\text{pts}} = 1500$ and $N_{\text{int}} = 100$.

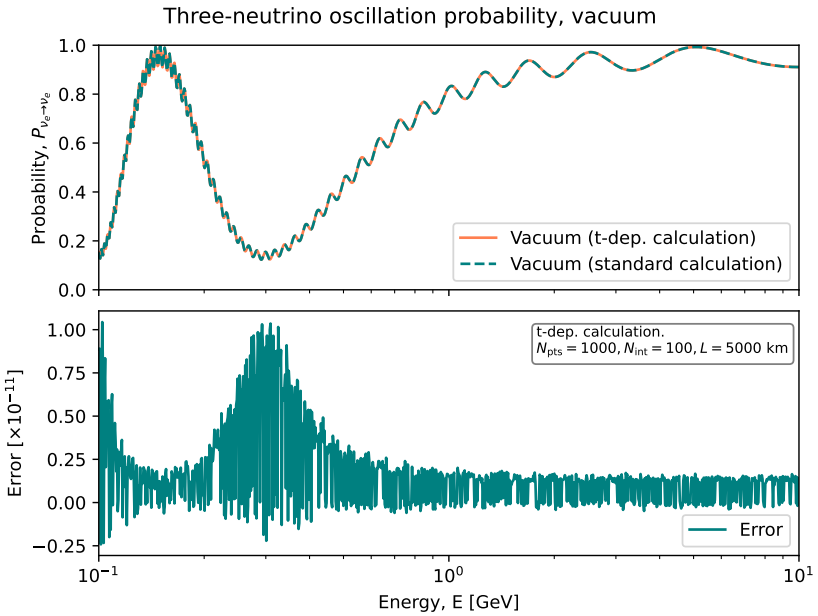

Figure 4: Three-neutrino survival probability computed using the methods introduced in Eqs. 27 and 29, as a function of energy in vacuum evaluated at $L = 5000$ km, comparing it to the probability in vacuum calculated using the standard formula (Eq. 54). The data consists of $N_{\text{pts}} = 1000$ and $N_{\text{int}} = 100$.

The fact that the errors in Figs. 1, 2, 3 and 4 are so small gives credibility to the new method.

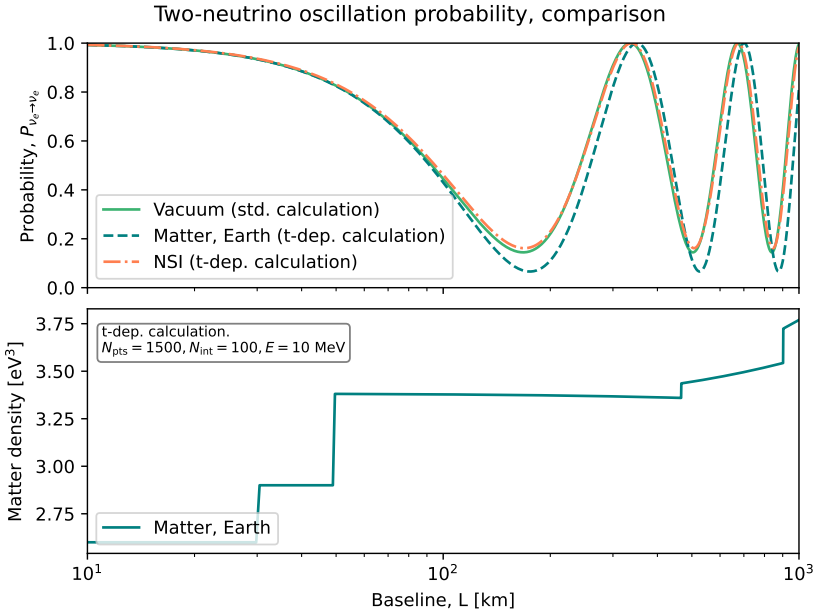

Figure 5: *Top:* Two-neutrino survival probability computed using the methods introduced in Eqs. 26 and 28, as a function of distance in matter for standard and non-standard neutrino interactions (NSI) with matter evaluated at $E = 10$ MeV, where the matter profile is the one of the Earth, comparing it to the probability calculated using the standard formula (Eq. 10). *Bottom:* Matter density inside the Earth according to the Preliminary Reference Earth Model (PREM) [2] as a function of distance. The data consists of $N_{\text{pts}} = 1500$ and $N_{\text{int}} = 100$.

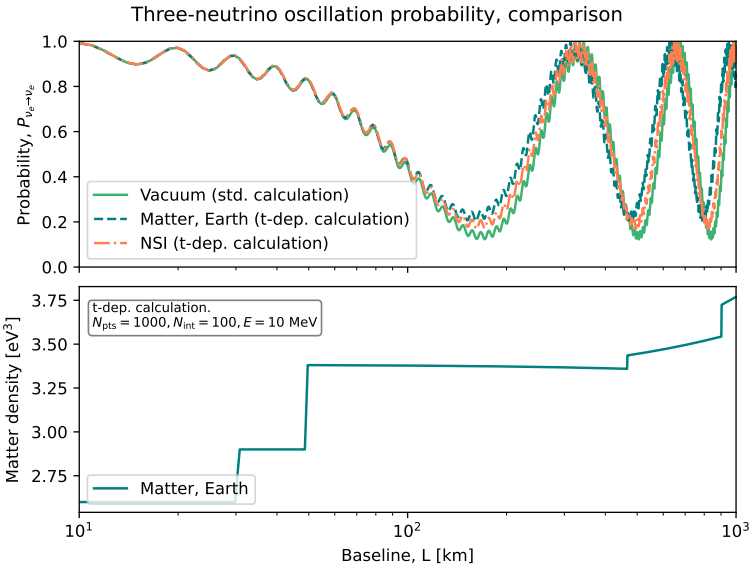

Figure 6: *Top:* Three-neutrino survival probability computed using the methods introduced in Eqs. 27 and 29, as a function of distance in matter for standard and non-standard neutrino interactions (NSI) with matter evaluated at $E = 10$ MeV, where the matter profile is the one of the Earth, comparing it to the probability calculated using the standard formula (Eq. 54). *Bottom:* Matter density inside the Earth according to the Preliminary Reference Earth Model (PREM) [2] as a function of distance. The data consists of $N_{\text{pts}} = 1000$ and $N_{\text{int}} = 100$.

4.2 Two- and Three-Neutrino Oscillations in Matter

We expect to see a difference from the neutrino oscillation probability in vacuum which is greater, the larger the matter density is.

Figures 5 and 6 show the probability as a function in matter for standard and non-standard neutrino interactions with matter and in vacuum using Eq. 10. For non-standard neutrino interactions with matter, mediated by a new neutral boson, in the two-flavor case we use

$$\mathbb{H}_2^{\text{NSI}} = \mathbb{H}_2^{\text{vac}} + V_W \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + V_W \begin{pmatrix} \epsilon_{ee} & \epsilon_{e\mu} \\ \epsilon_{e\mu}^* & \epsilon_{\mu\mu} \end{pmatrix}, \tag{31}$$

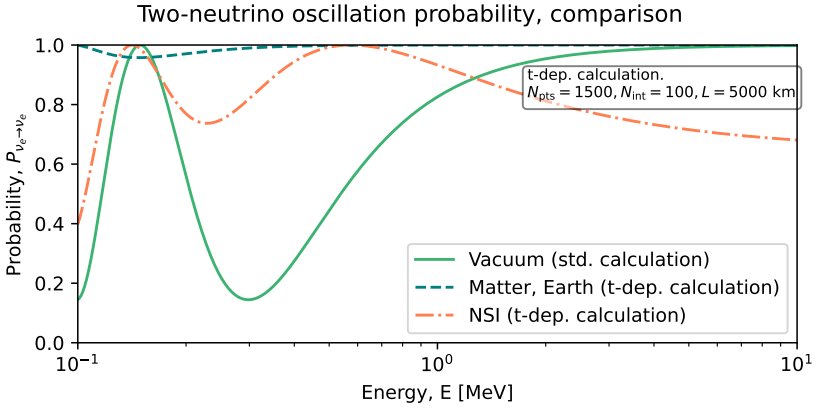

Figure 7: Two-neutrino survival probability computed using the methods introduced in Eqs. 26 and 28, as a function of energy in matter for standard and non-standard neutrino interactions (NSI) with matter evaluated at $L = 5000$ km, where the matter profile is the one of the Earth, comparing it to the probability calculated using the standard formula (Eq. 10). The data consists of $N_{\text{pts}} = 1500$ and $N_{\text{int}} = 100$.

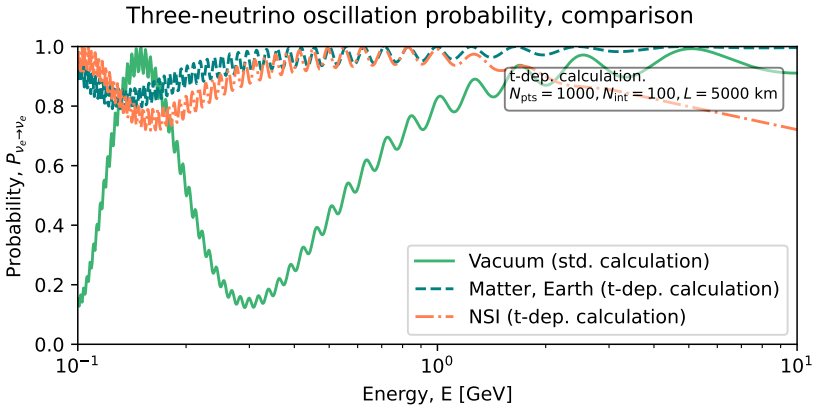

Figure 8: Three-neutrino survival probability computed using the methods introduced in Eqs. 27 and 29, as a function of energy in matter for standard and non-standard neutrino interactions (NSI) with matter evaluated at $L = 5000$ km, where the matter profile is the one of the Earth, comparing it to the probability calculated using the standard formula (Eq. 54). The data consists of $N_{\rm pts} = 1000$ and $N_{\rm int} = 100$.

and in the three-flavor case

$$\mathbb{H}_3^{\rm NSI} = \mathbb{H}_3^{\rm vac} + V_W \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + V_W \begin{pmatrix} \epsilon_{ee} & \epsilon_{e\mu} & \epsilon_{e\tau} \\ \epsilon_{e\mu}^* & \epsilon_{\mu\mu} & \epsilon_{\mu\tau} \\ \epsilon_{e\tau}^* & \epsilon_{\mu\tau}^* & \epsilon_{\tau\tau} \end{pmatrix}, \tag{32}$$

where the new neutrino couplings with matter (electrons and quarks) are defined conventionally [7, 8, 14, 18] and we set them to values that are approximately allowed by present-day data [7]: $\epsilon_{ee} = -\epsilon_{e\mu} = 0.06$, $\epsilon_{\mu\mu} = 1.2$ and $\epsilon_{e\tau} = \epsilon_{\mu\tau} = \epsilon_{\tau\tau} = 0$.

Figures 5 and 6 show that, as expected, matter effects are larger where the electron number density is larger. As in Fig. 2, we see that there is another layer of structure in Fig. 6 compared to Fig. 5.

Figures 7 and 8 show a similar behavior for the three-neutrino survival probability in vacuum as for the two-neutrino probability in Figs. 5 and 6. The error in our calculation compared to the standard expression are of the same order as before.

4.3 Two-Neutrino Resonant Oscillations in the Sun

The matter density in the Sun is

$$N_e = 245 \frac{N_A}{\rm cm^3} e^{\frac{-r}{r_0}}. \tag{33}$$

We introduce the effective mixing angle in matter

$$\sin(2\theta_M) = \frac{\tan(2\theta)}{1 - \frac{2EV_W}{\Delta m^2 \cos(2\theta)}}, \tag{34}$$

as well as the effective squared-mass difference

$$\Delta m_M^2 = \sqrt{(\Delta m^2 \cos(2\theta) - 2EV_W)^2 + (\Delta m^2 \sin(2\theta))^2}. \tag{35}$$

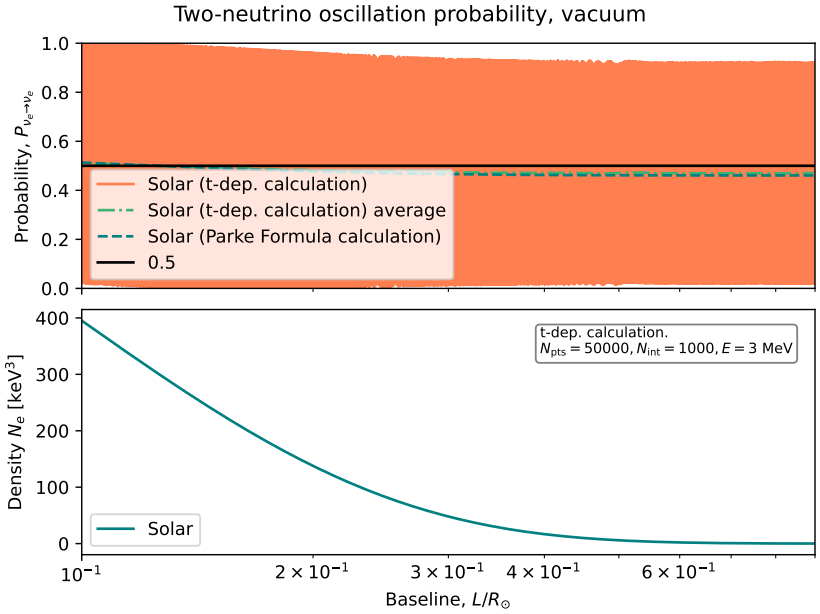

Figure 9: *Top:* Two-neutrino survival probability computed using the methods introduced in Eqs. 26 and 28, as a function of distance in matter evaluated at $E = 3$ MeV, where the matter profile is the one of the Sun, comparing it to the Parke formula (Eq. 37) [21]. *Bottom:* Matter density as a function of the distance. The data consists of $N_{\rm pts} = 50000$ and $N_{\rm int} = 1000$.

When the mixing is at its maximum, the number of neutrinos of one flavor transitioning to another flavor is at its maximum, i.e. a resonance happens. If the region where the resonance conditions are satisfied is large enough, it is possible that all these neutrinos change flavor. This is the Mikheyev–Smirnov–Wolfenstein (MSW) effect [13, 23]. At the resonance, the matter potential becomes $2EV_W = \Delta m^2 \cos(2\theta)$, and so, from Eqs. 34 and 35, the mixing angle and mass-squared difference become

$$\theta_M|_R = \frac{\pi}{4}, \Delta m^2_M|_R = \Delta m^2 \sin(2\theta)\,. \tag{36}$$

For resonant flavor transitions in the Sun, we have a good approximation for the neutrino oscillation probability in the two-flavor case given by the Parke formula [21],

$$\overline{P}_{\nu_e \to \nu_e} = \frac{1}{2} + \left(\frac{1}{2} - P_c\right) \cos 2\theta^{(i)}_M \cos 2\theta, \tag{37}$$

with $\theta^{(i)}_M$ being the effective mixing angle at the origin of the neutrinos, and P_c being the crossing probability, which is the probability of the neutrino mass eigenstates transitioning at the resonance [5, 21]. The Parke formula is an approximation of the average survival probability, derived under the conditions that the neutrinos are created at a large density, N_e, before the resonance, where N_e decreases such that $\theta^{(f)}_M \simeq \theta$, meaning that the neutrinos are detected in a medium very close to vacuum. These conditions are satisfied for the MeV-scale neutrinos born in the nuclear reactions inside the Sun.

Figure 9 shows the probability as a function in matter in the solar case with the potential

$$V_{solar} = \sqrt{2} G_F N_e = \sqrt{2} G_F 245 \frac{N_A}{\text{cm}^3} e^{\frac{-r}{r_0}}. \tag{38}$$

The average of neutrino survival probability follows the average given by the Parke formula.

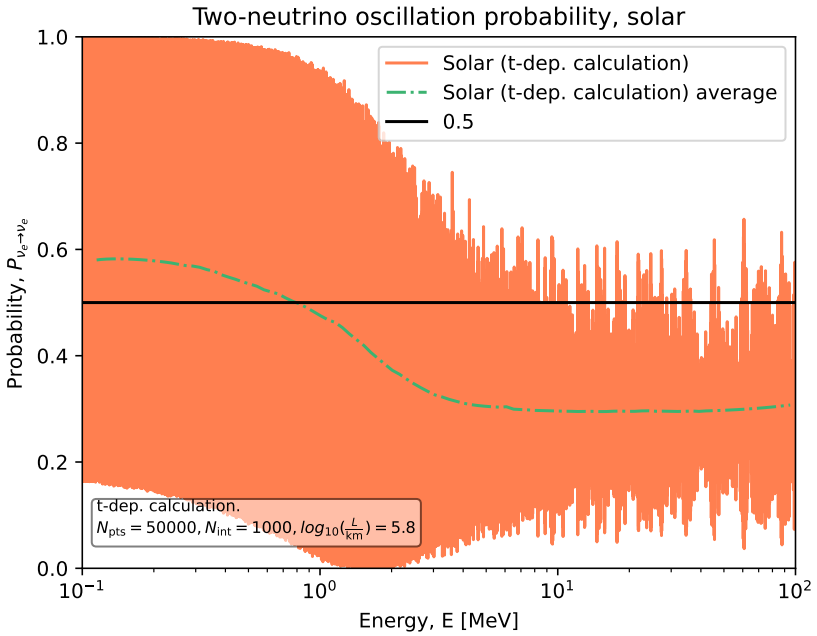

Figure 10: Two-neutrino survival probability computed using the methods introduced in Eqs. 26 and 28, as a function of energy in the two-flavor case in matter evaluated at $L = 10^6$ km, where the matter profile is the one of the Sun. The data consists of $N_{\text{pts}} = 50000$ and $N_{\text{int}} = 1000$.

Figure 10 shows the probability as a function in matter in the solar case with the potential from Eq. 38. Here we see the resonance, as expected, where the neutrino oscillation probability dips to zero at around $E \approx 15$ MeV, meaning that $\nu_e \to \nu_\mu$ transitions become very likely.

5 Summary and Outlook

We have introduced a new method to compute neutrino oscillation probabilities for arbitrary time-dependent Hamiltonians, for two-neutrino and three-neutrino systems. Our strategy has been to split wide neutrino baselines into multiple small intervals, ***within which we compute the time-evolution operators using exact methods***.

Figures 1, 2, 3 and 4 show the neutrino oscillation probability in vacuum computed with this new method, comparing it to the exact solution seen in Eqs. 10 and 54. The plots show that the method works very well with a small error in the order of $\sim 10^{-10}$. This result shows that the method is accurate for time-independent Hamiltonians, which gives great credibility to the method.

Using the newly developed method to compute the neutrino survival probability in matter, the plot shows that the probability acts as expected, but there is no exact solution to compare it to. In Figures 5, 6, 7 and 8 the potential of the earth has been used, and the probability has been found for standard interactions and NSI. They behave as expected. In Figures 9 and 10 the potential of the Sun has been used (Eq. 38) and the probability and the average of the probability has been found and compared to the average found via the Parke formula (Eq. 37). The average of the probability behave as predicted by the Parke formula, and we also see a resonance as expected.

In conclusion, this method is an easy and accurate way to compute neutrino oscillation probabilities with two- and three-neutrino flavors. If one were to do further work on this project, it would be interesting to test the method against other methods.

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Appendix

A: Classical Derivation of the Two-Neutrino Oscillation Probability in Matter

In the two-flavor approximation, the mixing matrix is defined as

$$\mathbb{U}_\theta = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}. \tag{39}$$

The neutrino flavor eigenstate, $|\nu_\alpha\rangle$, is a superposition of the flavor eigenstates, $|\nu_\alpha\rangle$ and $|\nu_\beta\rangle$,

$$|\nu_\alpha\rangle = f_e|\nu_e\rangle + f_\mu|\nu_\mu\rangle, \tag{40}$$

where $|\nu_e\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|\nu_\mu\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, and f_e and f_μ are the amplitudes.

The flavor eigenstate is an eigenstate of the Hamiltonian in matter in the flavor space and we have the Schrödinger equation

$$i\frac{\partial}{\partial t}|\nu_\alpha\rangle = H^M|\nu_\alpha\rangle = E_\alpha|\nu_\alpha\rangle, \tag{41}$$

with the solution

$$|\nu_\alpha(t)\rangle = e^{-iH^M t}\left(f_e|\nu_e\rangle + f_\mu|\nu_\mu\rangle\right). \tag{42}$$

The Hamiltonian in matter will be equal to the Hamiltonian in vacuum added the potentials from the interactions with W and Z bosons

$$\begin{aligned} H^M &= H^{\text{vac}} + V_W \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + V_Z \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= H^{\text{vac}} + \frac{V_W}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \frac{V_W}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= H^{\text{vac}} + \frac{V_W}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \end{aligned} \tag{43}$$

where $V_Z \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\frac{V_W}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ are global phases, and where H^{vac} is the Hamiltonian in vacuum in the flavor space:

$$H^f = \mathbb{U}_\theta H^m \mathbb{U}_\theta^\dagger = \frac{\Delta m^2}{4E} \begin{pmatrix} -\cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & \cos(2\theta) \end{pmatrix} \equiv H^{\text{vac}}. \tag{44}$$

We can now find the Hamiltonian in matter in the flavor basis:

$$\begin{aligned} H^M &= H^{\text{vac}} + \frac{V_W}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ &= \frac{\Delta m^2}{4E} \begin{pmatrix} -\cos(2\theta) + x & \sin(2\theta) \\ \sin(2\theta) & \cos(2\theta) - x \end{pmatrix}, \end{aligned} \tag{45}$$

where $x = \frac{V_W/2}{\Delta m^2/4E}$. We then define $\Delta m_M^2 \equiv \Delta m^2\sqrt{\sin^2(2\theta) + (\cos(2\theta) - x)^2}$ and $\sin^2(2\theta_M) \equiv \frac{\sin^2(2\theta)}{\sin^2(2\theta)+(\cos(2\theta)-x)^2}$. I'll now be inserting first Δm_M^2 and then $\sin^2(2\theta_M)$:

$$H^M = \frac{\Delta m_M^2}{4E}\begin{pmatrix} \frac{-\cos(2\theta)+x}{\sqrt{\sin^2(2\theta)+(\cos(2\theta)-x)^2}} & \frac{\sin(2\theta)}{\sqrt{\sin^2(2\theta)+(\cos(2\theta)-x)^2}} \\ \frac{\sin(2\theta)}{\sqrt{\sin^2(2\theta)+(\cos(2\theta)-x)^2}} & \frac{\cos(2\theta)-x}{\sqrt{\sin^2(2\theta)+(\cos(2\theta)-x)^2}} \end{pmatrix} = \frac{\Delta m_M^2}{4E}\begin{pmatrix} -\cos(2\theta_M) & \sin^2(2\theta_M) \\ \cos(2\theta_M) & \sin^2(2\theta_M) \end{pmatrix}. \tag{46}$$

Here it is clear that $H^M \to H^{\text{vac}}$ when $\Delta m_M^2 \to \Delta m^2$ and $\theta_M \to \theta$. Therefore they must have the same solution.

B: Classical Derivation of the Three-Neutrino Oscillation Probability in Vacuum

Working with three flavors, the mixing matrix is defined as

$$\mathbb{U}_{\text{PMNS}} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta_{CP}} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta_{CP}} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta_{CP}} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta_{CP}} & -c_{12}s_{23} - s_{12}c_{23}s13e^{i\delta_{CP}} & c_{23}c_{13} \end{pmatrix}. \tag{47}$$

The neutrino flavor eigenstates, $|\nu_\alpha\rangle$, are superpositions of the mass eigenstates, $|\nu_i>$,

$$|\nu_\alpha\rangle = \sum_i \mathbb{U}^*_{\text{PMNS},\alpha i}|\nu_i\rangle. \tag{48}$$

The mass eigenstates are eigenstates of the Hamiltonian and se have the following Schrödinger equation

$$i\frac{\partial}{\partial t}|\nu_i\rangle = H|\nu_i\rangle = E_i|\nu_i\rangle, \tag{49}$$

with the solution

$$|\nu_i(t)\rangle = e^{-iE_i t}|\nu_i\rangle. \tag{50}$$

Since the neutrinos are relativistic, we can write the energy as

$$E_i = \sqrt{|\vec{p}|^2 + m_i^2} \simeq |\vec{p}| + \frac{m_i^2}{2|\vec{p}|} \simeq \frac{m_i^2}{2E}, \tag{51}$$

where it is used that the momentum is the same for all eigenstates and is therefore a global phase, and that neutrinos are relativistic so that $m_i << |\vec{p}|$ and $E \simeq |\vec{p}|$. In the mass basis, the Hamiltonian will be the diagonal of the energies:

$$H^m = \text{diag}(E_1, E_2, E_3) = \begin{pmatrix} E_1 & 0 & 0 \\ 0 & E_2 & 0 \\ 0 & 0 & E_3 \end{pmatrix} = \begin{pmatrix} \frac{m_1^2}{2E} & 0 & 0 \\ 0 & \frac{m_2^2}{2E} & 0 \\ 0 & 0 & \frac{m_3^2}{2E} \end{pmatrix}. \tag{52}$$

The amplitude and therefore the probability of the neutrino oscillation in the two-flavor approximation can now be found

$$\mathbb{U}_3(L) = \langle\nu_\beta|\nu_\alpha(t)\rangle = \sum_{i,j} U^*_{0,\alpha i}U_{0,\beta i}e^{-i\frac{m_i^2}{2E}t}\langle\nu_j|\nu_i\rangle = \sum_i U^*_{0,\alpha i}U_{0,\beta i}e^{-i\frac{m_i^2}{2E}t}. \tag{53}$$

The neutrino oscillation probability can now easily be calculated,

$$
\begin{aligned}
P_{\nu_\alpha \to \nu_\beta} &= |\mathbb{U}_3(L)|^2 \\
&= \sum_{i,j} U^*_{0,\alpha i} U_{0,\beta i} U_{0,\alpha j} U^*_{0,\beta j} e^{-i\frac{\Delta m^2_{ij}}{2E}L} \\
&= \delta_{\alpha\beta} - 4\sum_{i>j} \mathcal{R}\left(U^*_{0,\alpha i} U_{0,\beta i} U_{0,\alpha j} U^*_{0,\beta j}\right) \sin^2\left(\frac{\Delta m^2_{ij}}{2E}L\right) \\
&+ 2\sum_{i>j} \mathcal{I}\left(U^*_{0,\alpha i} U_{0,\beta i} U_{0,\alpha j} U^*_{0,\beta j}\right) \sin\left(\frac{\Delta m^2_{ij}}{2E}L\right),
\end{aligned}
\tag{54}
$$

which is then the neutrino oscillation probability.